Rigidity in special relativity

2004 J. Phys. A: Math. Gen. 37 4369

(http://iopscience.iop.org/0305-4470/37/15/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4369–4378 PII: S0305-4470(04)68844-5

Rigidity in special relativity

Dongho Kim and Sang Gyu Jo[1]

Department of Physics, Kyungpook National University, Daegu 702-701, Korea

E-mail: sgjo@knu.ac.kr

Received 14 September 2003, in final form 17 February 2004
Published 29 March 2004
Online at stacks.iop.org/JPhysA/37/4369 (DOI: 10.1088/0305-4470/37/15/006)

Abstract
We discuss the finite version of rigid motions in special relativity. Focusing on the extension of the size of a rigid motion, we investigate the reflexive and transitive properties of rigid motions. We show that rigid rotation of a three-dimensional object is impossible while its rigid translation is possible. It is shown that a two-dimensional plane can rotate rigidly.

PACS number: 03.30.+p

1. Introduction

Before the special theory of relativity was known, time was thought to be absolutely separated from space and velocity of a particle was believed to have no upper bound. Hence, even the velocity of transmission of a force signal was assumed to be infinite. This assumption makes possible the notion of a rigid body, which plays an important role in classical physics. A rigid body is defined as a collection of particles whose relative distances are constrained to remain absolutely fixed. If we pull a particle in a rigid body, all the other particles should react instantaneously in order to maintain the relative distances. This is possible only when force signals are transmitted with an infinite velocity, which does not conflict with classical physics.

By contrast, the special theory of relativity, which Albert Einstein established in 1905 [1], does not allow the concept of a rigid body. According to this theory, not only the velocity of a particle but also that of transmission of a force signal should not exceed the speed of light. If, for example, we pull on one end of a straight rod which is 300 million meters long, it will take at least one second for the other end to receive a force signal. The other end would not move for one second, and the length of the rod cannot be fixed. This fact does not change, although the mass of the rod is very small or the rod is formed tremendously firmly. As a result, the concept of a rigid body may not exist in special relativity.

Nevertheless, there are reasons why we discuss rigidity. We are so much accustomed to the concept of rigidity that it is natural to search for possibilities of holding the concept

[1] Corresponding author.

in a relativistic way. There have been several versions of relativistic rigid motion among which Born's definition is most popularly used [2–8]. His definition deals with infinitesimal spatial length elements to accommodate general relativity. We consider here only special relativistic rigid motions. A motion of a system of particles is defined to be a rigid motion if the distance between any two particles in the system remains the same throughout the motion when observed in an instantaneous rest inertial frame with respect to any one of the two particles. Note here that distances are not Lorentz-invariant quantities. Therefore, when we compare distances, we have to specify inertial frames to work on. By the definition, an object at rest in an inertial frame is observed to move rigidly by a uniformly moving observer in spite of the length contraction.

We emphasize here that an appropriate force should be given to each particle so that the motion remains to be rigid. This is why we use 'rigid motion' rather than 'rigid body'. A rigid body would be an object which moves rigidly, as described above, regardless of the way forces are applied, and we know such thing cannot exist owing to the finiteness of signal velocity. The notion of rigid motion does not care about the finiteness of signal velocity.

Now the question is how far we can go with our definition of a rigid motion. In other words, are there such motions at all? We will see that there exist such motions but not as much as in classical physics. As in classical physics, we describe rigid motions by two types of motions: a rigid translation and a rigid rotation. It turns out that a three-dimensional object can undergo a one-dimensional rigid translation. However, as was predicted by Ehrenfest's paradox [9], a rigid rotation of a three-dimensional object is shown to be impossible. Therefore rigid motions are very much restricted in special relativity.

In section 2, we discuss a one-dimensional rigid translation. Here we derive an equation which governs the motion of each particle and introduce a hyperbolic rigid motion as an example of a rigid translation. In section 3, we discuss a rigid rotation. Here we assume one point particle is fixed while the other particles are rotating around the one. We derive a set of equations for rigid rotations and discuss possibilities of rigid rotations. Finally, we give a summary in conclusion.

2. Rigid translation in special relativity

2.1. Equation of rigid translation

In this section, we consider a one-dimensional translational motion. For simplicity we begin with a motion of two particles. Initially, they are at rest. One(particle A) is at x_0, and the other (particle B) is at $x_0 - L_0$. The distance between the two particles is L_0. Now, at $t = 0$, the particle A begins to move by a given pattern $x_A = x_A(t)$. What is the corresponding motion of the particle B, $x_B = x_B(t)$, which maintains the rigidity of our two particle system? According to classical physics, the answer to this question would be $x_B(t) = x_A(t) - L_0$, and the velocities of two particles would satisfy $\dot{x}_B(t) = \dot{x}_A(t)$. However, according to special theory of relativity, the length of a moving object should be shorter by an observer at rest.

We now give a natural definition of a rigid motion from a special relativistic viewpoint. A motion of a system of particles is defined to be a rigid motion if the distance between any two particles in the system remains the same throughout the motion when observed in an instantaneous rest inertial frame with respect to any one of the two particles. If we apply this definition of a rigid motion to our one-dimensional rigid translation, the answer to the above question is as follows. The motion of particle B should be in such a way that particle A must be L_0 ahead of particle B at any moment in an instantaneous rest frame of particle A. Let us formulate this statement.

Let frame O_0 be the rest frame for both particles before they begin to move. Their motions $x_A(t)$ and $x_B(t)$ are coordinatized in this frame. Let frame O_t be the instantaneous rest frame with respect to particle A when time is t. When $t = 0$, frame O_0 and frame O_t coincide. Furthermore, the relative velocity of frame O_t with respect to frame O_0 is $\dot{x}_A(t)$. In frame O_t, particle A is instantaneously at rest. In this frame, particle B should be exactly $-L_0$ from the location of particle A at equal time in order to satisfy the definition of a rigid translation. By Lorentz-transforming back to frame O_0, we get the position of particle B in frame O_0,

$$\begin{pmatrix} ct \\ x_A(t) \end{pmatrix} + \begin{pmatrix} \gamma_A(t) & \gamma_A(t)\beta_A(t) \\ \gamma_A(t)\beta_A(t) & \gamma_A(t) \end{pmatrix} \begin{pmatrix} 0 \\ -L_0 \end{pmatrix} = \begin{pmatrix} ct - \gamma_A(t)\beta_A(t)L_0 \\ x_A(t) - \gamma_A(t)L_0 \end{pmatrix} \tag{1}$$

where $\beta_A(t) = \frac{\dot{x}_A(t)}{c}$ and $\gamma_A(t) = \frac{1}{\sqrt{1-\beta_A^2(t)}}$. This forms the worldline of particle B, and $x_B(t)$ must satisfy the following equation:

$$x_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right) = x_A(t) - \gamma_A(t)L_0. \tag{2}$$

This equation is the equation of rigid translation. In the derivation of this equation, we used the fact that the distance between the two particles is L_0 in an instantaneous rest frame with respect to particle A. We also have to check that the distance between the two particles is L_0 in an instantaneous rest frame with respect to particle B. The corresponding equation is as follows:

$$x_A\left(t + \frac{\gamma_B(t)\beta_B(t)L_0}{c}\right) = x_B(t) + \gamma_B(t)L_0. \tag{3}$$

If this equation and (2) are compatible, we say that they satisfy the reflexive property.

2.2. *Reflexive property and transitive property*

From (2), we can find the velocity of particle B. Differentiating both sides of this equation with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right)\right] = \dot{x}_A(t) - \dot{\gamma}_A(t)L_0. \tag{4}$$

The first term on the right-hand side is equal to $c\beta_A(t)$, and $\dot{\gamma}_A(t)$ in the second term is

$$\dot{\gamma}_A(t) = \gamma_A^3(t)\beta_A(t)\dot{\beta}_A(t). \tag{5}$$

Hence, the right-hand side of (4) becomes

$$\dot{x}_A(t) - \dot{\gamma}_A(t)L_0 = \dot{x}_A(t)\left(1 - \frac{\gamma_A^3(t)\dot{\beta}_A(t)L_0}{c}\right). \tag{6}$$

Using the chain rule, the left-hand side of (4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right)\right] = \left(1 - \frac{\gamma_A^3(t)\dot{\beta}_A(t)L_0}{c}\right)\dot{x}_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right). \tag{7}$$

From (6) and (7), we get

$$\dot{x}_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right) = \dot{x}_A(t). \tag{8}$$

This equation implies that particle B is also instantaneously at rest in an instantaneous rest frame of particle A. Now, we are ready to prove the reflexive property. We assume (2) is satisfied for all t. Let

$$t - \frac{\gamma_A(t)\beta_A(t)L_0}{c} = \tau$$

then

$$t = \tau + \frac{\gamma_A(t)\beta_A(t)L_0}{c}$$

and (2) becomes

$$x_B(\tau) = x_A\left(\tau + \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right) - \gamma_A(t)L_0. \tag{9}$$

From (8), $\beta_A(t) = \beta_B(\tau)$ and $\gamma_A(t) = \gamma_B(\tau)$. Therefore, we get

$$x_A\left(\tau + \frac{\gamma_B(\tau)\beta_B(\tau)L_0}{c}\right) = x_B(\tau) + \gamma_B(\tau)L_0 \tag{10}$$

and this is the same as (3) with parameter t substituted by parameter τ.

We now consider the transitive property. Assume particle A and particle B are in a rigid translation and particle B and particle C are also in a rigid translation. We have the following relations:

$$x_A\left(\tau + \frac{\gamma_B(\tau)\beta_B(\tau)L_{AB}}{c}\right) = x_B(\tau) + \gamma_B(\tau)L_{AB} \tag{11}$$

$$x_B\left(t + \frac{\gamma_C(t)\beta_C(t)L_{BC}}{c}\right) = x_C(t) + \gamma_C(t)L_{BC}. \tag{12}$$

The distance between particle A and particle B is denoted by L_{AB} while the distance between particle B and particle C is denoted by L_{BC}. Note that these two equations are satisfied for any t and τ.

We want to prove that the motion of particle A and particle C is a rigid translation. This fact would be given by the following equation:

$$x_A\left(t + \frac{\gamma_C(t)\beta_C(t)L_{AC}}{c}\right) = x_C(t) + \gamma_C(t)L_{AC}. \tag{13}$$

Here, $L_{AC} = L_{AB} + L_{BC}$. To prove (13) we let $\tau = t + \frac{\gamma_C(t)\beta_C(t)L_{BC}}{c}$ in (12). Differentiating (12) with respect to t and dividing the common factor as in the derivation of (8), we obtain $\beta_B(\tau) = \beta_C(t)$ and $\gamma_B(\tau) = \gamma_C(t)$. The left-hand side of (11) can be written as

$$\begin{aligned} x_A\left(\tau + \frac{\gamma_B(\tau)\beta_B(\tau)L_{AB}}{c}\right) &= x_A\left(t + \frac{\gamma_C(t)\beta_C(t)L_{BC}}{c} + \frac{\gamma_B(\tau)\beta_B(\tau)L_{AB}}{c}\right) \\ &= x_A\left(t + \frac{\gamma_C(t)\beta_C(t)L_{AC}}{c}\right) \end{aligned} \tag{14}$$

while the right-hand side of (11) can be written as follows:

$$x_B(\tau) + \gamma_B(\tau)L_{AB} = x_C(t) + \gamma_C(t)L_{BC} + \gamma_B(\tau)L_{AB} = x_C(t) + \gamma_C(t)L_{AC}. \tag{15}$$

We used (12) to get (15). It is clear that (14) and (15) imply (13), and the proof is completed.

In fact, there is an easier way to prove the transitive property of rigid translations. We know that if the motion of particle A and particle B is a rigid translation, they share an instantaneous rest frame. Similarly, if the motion of particle B and particle C is a rigid translation, these two particles share an instantaneous rest frame. The instantaneous rest frame with respect to particle B is unique at every moment and we conclude that at every moment the three particles share only one instantaneous rest frame. Furthermore, the distance between particle A and particle C is the sum of L_{AB} and L_{BC} measured in the rest frame. Consequently the motion of particle A and particle C is also a rigid translation.

The transitive property of rigid translations allows an extended object to move rigidly. We substitute B and L_0 in (2) by a continuous parameter h,

$$x_h\left(t - \frac{\gamma_A(t)\beta_A(t)h}{c}\right) = x_A(t) - \gamma_A(t)h. \tag{16}$$

This equation describes a rigid translation of a one-dimensional rod parametrized by h. The end point x_A corresponds to x_h with $h = 0$. We introduce an example of a rigid translation of a rod in the next subsection.

2.3. Hyperbolic rigid motion

If particle A undergos a hyperbolic motion given by

$$x_A(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right) \tag{17}$$

with a constant a, which is the acceleration of particle A, then (16) becomes

$$x_h\left(t - \frac{ath}{c^2}\right) = x_A(t) - \gamma_A(t)h. \tag{18}$$

This can be solved easily and we get

$$x_h(t) = \frac{c^2}{a}\left[\sqrt{\left(1 - \frac{ah}{c^2}\right)^2 + \frac{a^2t^2}{c^2}} - 1\right] \tag{19}$$

which can be rearranged as

$$x_h(t) = \frac{c^2}{\frac{a}{1-ah/c^2}}\left[\sqrt{1 + \left(\frac{a}{1-ah/c^2}\right)^2\frac{t^2}{c^2}} - 1\right] - h. \tag{20}$$

Comparing this equation with (17), we see that the particle parametrized by h is in a hyperbolic motion with acceleration $\frac{a}{1-ah/c^2}$. From the finiteness of acceleration, we get a restriction for h by $h < c^2/a$. The velocity of each particle in the rod is obtained by differentiating (20) with respect to t,

$$\beta_h(t) = \frac{\left(\frac{a}{1-ah/c^2}\right)\frac{t}{c}}{\sqrt{1 + \left(\frac{a}{1-ah/c^2}\right)^2\frac{t^2}{c^2}}}. \tag{21}$$

We notice, from the above two equations, that at $t = 0$ every particle is at rest and h is the distance of the particle parametrized by h from particle A.

3. Rigid rotation in special relativity

3.1. Rigid rotation of three particles

Consider a system of two particles, particle A and particle O. Particle O is fixed at the origin of an inertial frame O_0 and particle A is rotating around this point on the x–y plane with a radius a. In frame O_0, the distance between the two particles is fixed to be a, and the motion is rigid by the viewpoint of particle O. Under a boost transformation of a given relative velocity, distances orthogonal to the velocity do not change. Therefore the distance between the two particles observed in an instantaneous rest inertial frame with respect to particle A is also fixed to be a. We call this kind of rigid motion a 'rigid rotation'.

Now we introduce one more particle, particle B, rotating around particle O on the same plane with a radius b. What concerns us is the following question. When particle A rotates around particle O with a given pattern by $\theta_A(t)$, how should particle B rotate around particle O so that the three particles move rigidly? We investigate an answer to this question now.

Using the freedom in choosing space axes, we let the x-axis pass through particle A at an arbitrarily given time t_0 in frame O_0. The worldline of particle A, parametrized by t, is given by

$$\begin{pmatrix} ct \\ a\cos[\theta_A(t)-\theta_A(t_0)] \\ a\sin[\theta_A(t)-\theta_A(t_0)] \end{pmatrix} \tag{22}$$

while the worldline of particle B is given by

$$\begin{pmatrix} ct \\ b\cos[\theta_B(t)-\theta_A(t_0)] \\ b\sin[\theta_B(t)-\theta_A(t_0)] \end{pmatrix} \tag{23}$$

where we suppressed the z-coordinate for simplicity. The velocity of particle A at time t_0 is $a\dot{\theta}_A(t_0)\hat{y}$. Therefore, in an instantaneous rest frame $F_A(t_0)$ with respect to particle A at time t_0, the worldline of particle A is described by

$$\begin{pmatrix} \gamma & 0 & -\gamma\beta \\ 0 & 1 & 0 \\ -\gamma\beta & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ a\cos[\theta_A(t)-\theta_A(t_0)] \\ a\sin[\theta_A(t)-\theta_A(t_0)] \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta a\sin[\theta_A(t)-\theta_A(t_0)] \\ a\cos[\theta_A(t)-\theta_A(t_0)] \\ -\gamma\beta ct + \gamma a\sin[\theta_A(t)-\theta_A(t_0)] \end{pmatrix} \tag{24}$$

while the worldline of particle B in frame $F_A(t_0)$ is described by

$$\begin{pmatrix} \gamma & 0 & -\gamma\beta \\ 0 & 1 & 0 \\ -\gamma\beta & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ b\cos[\theta_B(t)-\theta_A(t_0)] \\ b\sin[\theta_B(t)-\theta_A(t_0)] \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta b\sin[\theta_B(t)-\theta_A(t_0)] \\ b\cos[\theta_B(t)-\theta_A(t_0)] \\ -\gamma\beta ct + \gamma b\sin[\theta_B(t)-\theta_A(t_0)] \end{pmatrix} \tag{25}$$

where

$$\beta = \frac{a\dot{\theta}_A(t_0)}{c} \tag{26}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{27}$$

When $t = t_0$, particle A is instantaneously at rest in frame $F_A(t_0)$ and its spacetime coordinates are $(\gamma ct_0, a, -\gamma\beta ct_0)$. We need to know the location of particle B at this instant in frame $F_A(t_0)$. For that matter we have to identify the event corresponding to this instant by equating the time components of the two particles,

$$\gamma ct - \gamma\beta b\sin[\theta_B(t)-\theta_A(t_0)] = \gamma ct_0. \tag{28}$$

Using (26), the above equation can be written as

$$t - t_0 = \frac{ab\dot{\theta}_A(t_0)}{c^2}\sin[\theta_B(t) - \theta_A(t_0)]. \tag{29}$$

This is the equation of simultaneity stating that the event of particle B at time t and the event of particle A at time t_0 occur at the same instant in frame $F_A(t_0)$. Now, we impose the condition of rigidity, by requiring that the distance between particle A and particle B is fixed at this instant in frame $F_A(t_0)$. This condition can be expressed as

$$\{b\cos[\theta_B(t) - \theta_A(t_0)] - a\}^2 + \{\gamma\beta c(t_0 - t) + \gamma b\sin[\theta_B(t) - \theta_A(t_0)]\}^2 = a^2 + b^2 - 2ab\cos\phi_{AB} \tag{30}$$

where ϕ_{AB} is the angle formed by two radii a and b when the particles are at rest. Using (29), the above equation can be simplified further,

$$\frac{c^2}{2ab}(t - t_0)^2 + \cos[\theta_B(t) - \theta_A(t_0)] = \cos\phi_{AB}. \tag{31}$$

These two equations, (29) and (31), serve as the main equations for a rigid rotation. Given $\theta_A(t)$, $\theta_B(t)$ satisfying these equations is uniquely fixed and describes the rotation of particle B so that particle B is always at the same distance form particle A in an instantaneous rest frame of particle A. What about the distance of particle A from particle B observed in an instantaneous rest frame with respect to particle B? We now move on to the question regarding the reflexive property and the transitive property of rigid rotation.

3.2. *Reflexive property of rigid rotation*

We first note that the choice of t_0 was arbitrary at the beginning. Once t_0 is chosen, then the two equations, (29) and (31), determine the values of t and $\theta_B(t)$. Let us rename t_0 as t_A and the solution t as t_B. Then, the two equations are written as

$$t_B - t_A = \frac{ab\dot{\theta}_A(t_A)}{c^2}\sin[\theta_B(t_B) - \theta_A(t_A)] \tag{32}$$

$$\frac{c^2}{2ab}(t_B - t_A)^2 + \cos[\theta_B(t_B) - \theta_A(t_A)] = \cos\phi_{AB}. \tag{33}$$

The reflexive property of rigid rotations will indicate that the following two equations are implied by the above two equations with a suitable choice of $\tilde{t}_A$:

$$\tilde{t}_A - t_B = \frac{ba\dot{\theta}_B(t_B)}{c^2}\sin[\theta_A(\tilde{t}_A) - \theta_B(t_B)] \tag{34}$$

$$\frac{c^2}{2ba}(\tilde{t}_A - t_B)^2 + \cos[\theta_A(\tilde{t}_A) - \theta_B(t_B)] = \cos\phi_{AB}. \tag{35}$$

These equations are derived by requiring that the event of particle B at time t_B and the event of particle A at time $\tilde{t}_A$ occur at the same instant in frame $F_B(t_B)$, which is an instantaneous rest frame with respect to particle B at time t_B, and also that the distance between particle A and particle B is fixed at this instant in the same inertial frame. Note that these two equations are obtained from (32) and (33) by interchanging radii a and b, and subscripts A and B.

We will show that $\tilde{t}_A = t_A$ solves (34) and (35). Before showing that, let us first prove the following:

$$\dot{\theta}_B(t_B) = \dot{\theta}_A(t_A). \tag{36}$$

Differentiating (33) with respect to time t_A and rearranging it, we obtain

$$\frac{\mathrm{d}t_B}{\mathrm{d}t_A}\left\{\frac{c^2}{ab}(t_B - t_A) - \dot{\theta}_B(t_B)\sin[\theta_B(t_B) - \theta_A(t_A)]\right\}$$
$$-\left\{\frac{c^2}{ab}(t_B - t_A) - \dot{\theta}_A(t_A)\sin[\theta_B(t_B) - \theta_A(t_A)]\right\} = 0. \tag{37}$$

The second term on the left-hand side is zero by (32). Since $\mathrm{d}t_B/\mathrm{d}t_A$ is not zero, we get

$$\frac{c^2}{ab}(t_B - t_A) - \dot{\theta}_B(t)\sin[\theta_B(t_B) - \theta_A(t_A)] = 0 \tag{38}$$

which, with the help of (32), implies (36).

Substituting $\dot{\theta}_B(t_B)$ and $\tilde{t}_A$ in (34) and (35) by $\dot{\theta}_A(t_A)$ and t_A respectively, (34) and (35) become identical to (32) and (33). Therefore, the reflexive property holds in a rigid rotation. Note that we can choose (32) and (36), instead of (32) and (33), as the conditions for a rigid rotation. It is easy to show that (32) and (36) imply (33). Note also that if $\dot{\theta}_A(t)$ is a monotonically increasing function of time, then $\dot{\theta}_B(t)$ should also be a monotonically increasing function of time owing to (36). This is an important aspect which we will use later.

A special case is when two particles A and B are on the same line passing through the centre. In this case, ϕ_{AB} is either 0 or π. For any given $\theta_A(t)$, $\theta_B(t)$ becomes

$$\theta_B(t_B) = \theta_A(t_A) + \phi_{AB} \tag{39}$$

with $t_B = t_A$. This solves the rigidity conditions, (32) and (36), and the corresponding motion is a usual classical rigid motion with the same angular velocity.

3.3. Check of transitive property of rigid rotation

The question of the transitive property of rigid rotation is phrased as follows. Three particles (particle A, particle B and particle D) are rotating around particle O on a plane. When the system AB (composed of particle A, particle B and particle O) and the system BD (composed of particle B, particle D and particle O) are separately under rigid rotations, then is the system AD (composed of particle A, particle D and particle O) automatically under a rigid rotation? The answer to this question in classical physics is yes as in the case of rigid translation. However, in special relativity it turns out to be no as we show now.

We assume that particle A is slowly accelerating from the rest in such a way that $\dot{\theta}_A(t)$ is a monotonically increasing function of time. The fact that two particles A and B are under a rigid rotation implies the existence of t_B and $\theta_B(t)$ (for given t_A and $\theta_A(t)$) satisfying the following two equations:

$$\dot{\theta}_B(t_B) = \dot{\theta}_A(t_A) \tag{40}$$

$$t_B - t_A = \frac{ab\dot{\theta}_A(t_A)}{c^2}\sin[\theta_B(t_B) - \theta_A(t_A)]. \tag{41}$$

Here, note that $\theta_B(t)$ is uniquely fixed and $\dot{\theta}_B(t)$ is also a monotonically increasing function of time. Similarly, the fact that two particles B and D are under a rigid rotation implies the existence of t_D and $\theta_D(t)$ (for given t_B and $\theta_B(t)$) satisfying the following two equations:

$$\dot{\theta}_D(t_D) = \dot{\theta}_B(t_B) \tag{42}$$

$$t_D - t_B = \frac{bd\dot{\theta}_B(t_B)}{c^2}\sin[\theta_D(t_D) - \theta_B(t_B)]. \tag{43}$$

Here again, note that $\theta_D(t)$ is uniquely fixed and $\dot{\theta}_D(t)$ is also a monotonically increasing function of time. Now we have to ask if there is a solution $\tilde{t}_D$ satisfying the following two equations:

$$\dot{\theta}_D(\tilde{t}_D) = \dot{\theta}_A(t_A) \tag{44}$$

$$\tilde{t}_D - t_A = \frac{ad\dot{\theta}_A(t_A)}{c^2}\sin[\theta_D(\tilde{t}_D) - \theta_A(t_A)]. \tag{45}$$

Equations (40) and (42) imply

$$\dot{\theta}_D(t_D) = \dot{\theta}_A(t_A). \tag{46}$$

Because $\dot{\theta}_D(t)$ is a monotonically increasing function of time, the only solution of (44) is $\tilde{t}_D = t_D$. Therefore, the transitive property of rigid rotation is guaranteed if (45) is satisfied by $\tilde{t}_D = t_D$:

$$t_D - t_A = \frac{ad\dot{\theta}_A(t_A)}{c^2}\sin[\theta_D(t_D) - \theta_A(t_A)]. \tag{47}$$

Summing (41) and (43) and using (40), we obtain

$$t_D - t_A = \frac{\dot{\theta}_A(t_A)}{c^2}\{ab\sin[\theta_B(t_B) - \theta_A(t_A)] + bd\sin[\theta_D(t_D) - \theta_B(t_B)]\}. \tag{48}$$

Comparing (47) and (48), we observe that (47) is satisfied if

$$ab\sin[\theta_B(t_B) - \theta_A(t_A)] + bd\sin[\theta_D(t_D) - \theta_B(t_B)] = ad\sin[\theta_D(t_D) - \theta_A(t_A)] \tag{49}$$

is satisfied. This equation is not satisfied automatically, but serves as an additional condition for the rotation to be rigid. If this equation is satisfied, then the system of four particles is under a rigid rotation. We will analyse this equation in the next subsection.

3.4. Incompatibility of rigid rotation and special relativity

We observe from (49) that a one-dimensional rod can rotate rigidly about an axis intersecting perpendicularly to the rod. This is the case where either all of the angular functions $\theta_A(t)$, $\theta_B(t)$, and $\theta_D(t)$ are the same or one of them differs from the others by π. In this case, by equating $t_A = t_B = t_C$ we see that (40)–(43) are all satisfied. Furthermore, (49) is also satisfied for any values of radii a, b, d. Therefore, the motion of a one-dimensional rod rotating rigidly, with a single angular function $\theta(t)$, about an axis intersecting perpendicularly to itself, is a rigid rotation.

Now we consider a case when two of the three particles, particle A and particle B, are on the same line with $a \neq b$. These two particles are moving rigidly. Obviously, $\theta_A(t_A) = \theta_B(t_B)$ or $\theta_A(t_A) = \theta_B(t_B) + \pi$ with $t_A = t_B$. We choose $\theta_A(t_A) = \theta_B(t_B)$ for simplicity. This is the case where particle A and particle B are on one side of the line with respect to particle O. We assume particle D, not on the line but on the same plane orthogonal to the axis of rotation, is rotating rigidly with particle B. Therefore, (40)–(43) are satisfied. In order to see if particle A and particle D are rigid with respect to each other, (49) should be checked. This equation can be written as

$$bd\sin[\theta_D(t_D) - \theta_A(t_A)] = ad\sin[\theta_D(t_D) - \theta_A(t_A)] \tag{50}$$

which can never be satisfied for any value of $d > 0$. Note that if $\theta_D(t_D) = \theta_A(t_A)$, then $t_D = t_A$ and particle D should be on the same line as the other two particles, which contradicts our assumption. Therefore, the transitive property does not hold in general for a rigid rotation. We now summarize our results.

4. Conclusion

We have shown that a one-dimensional rod can move rigidly along the direction of the length. The dimensionality can be extended in such a way that the rigidity is maintained. Distances orthogonal to the direction of motion remain unchanged under a Lorenz transformation to an instantaneous rest frame. Therefore, a motion of a three-dimensional object is a rigid translation if the y-, z-coordinates of each constituent particle are fixed and for any pair of two particles their x-coordinates satisfy (2).

Similarly, the dimensionality of a rigidly rotating object can be extended to two. Consider a finite-sized plane through which the z-axis passes. If it rotates around the z-axis rigidly in a classical sense, then its motion is a rigid rotation even in special relativity. Unlike the translational case, extension to a three-dimensional object under a rigid rotation is impossible because of the violation of the transitive property as shown in the previous section. We, therefore, conclude that rigid motions are very much restricted in special relativity.

Acknowledgment

This research was supported by Kyungpook National University Research Team Fund, 2003.

References

[1] Einstien A 1905 *Ann. Phys., Lpz.* **17** 891
[2] Born M 1909 *Ann. Phys., Lpz.* **30** 1
[3] Herglotz G 1910 *Ann. Phys., Lpz.* **31** 393
[4] Noether F 1910 *Ann. Phys., Lpz.* **31** 919
[5] Hill E L 1946 *Phys. Rev.* **69** 488
[6] Fokker A D 1949 *Rev. Mod. Phys.* **21** 406
[7] Salzman G and Taub A H 1954 *Phys. Rev.* **95** 1659
[8] Bona C 1983 *Phys. Rev. D* **27** 1243
[9] Ehrenfest P 1909 *Phys. Z.* **10** 918